Three-place Identity

A companion piece to *Membership and Identity*

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ABSTRACT. In this paper it will be shown that all of mathematics can be expressed in terms of *relative identity* when this concept is formalized as a three-place predicate. My focus here will be on the *proof* of this theorem, though I'll also take a brief look at how three-place identity might help to expand the horizons of science, which is the main topic of a longer paper, *Membership and Identity*. [1]

Section 1. Axiom systems

Axiom systems have two roles in mathematics.

Their more familiar role is that of formalizing a branch of mathematics such as geometry or set theory. In this role an *axiom* is a statement that is *axiomatic*, i.e., that asserts something generally regarded as *self-evident*. Of course there is usually room for argument about what is or is not self-evident, so the same branch of mathematics may be axiomatized in different ways, as has happened in the case of geometry and set theory. In Section 1 of *Membership and Identity,* (abbreviated M&I), we'll take an extended look at set theory as axiomatized by Zermelo and Fraenkel at the turn of the twentieth century, an axiom system that many logicians and mathematicians today regard as the universal language of mathematics. Actually, this system, abbreviated ZF, itself comes in several versions since what is *really* axiomatic about sets is still controversial, though fortunately the details of this controversy needn't concern us here.

Their second role is that of defining mathematical *concepts.* Axioms in this role are neither true nor false since they do not assert anything; rather they "shape" the meaning of some predicate or term. Consider, for instance, the concept of *equality* as expressed by the *equality predicate* x=y. This predicate of course occurs in every branch of mathematics, pure or applied, but as a *concept* it is defined by a *stand-alone* axiom system with the following three axioms:

> Equality axiom 1) x=x
> Equality axiom 2) if x=y then y=x
> Equality axiom 3) if x=y and y=z then x=z.

As we'll see in Section 3 of M&I, every axiom system has a unique equality predicate called its *identity*, which, as the logician Quine has shown, is well-defined even in those axiom systems where it is not among the primitive concepts. [2]

The distinction between these two roles is not always clear-cut. We could, for instance, think of the equality axioms as self-evident truths about a concept that we already understand, thereby giving us a true *theory* of equality. On the other hand, we could think of set theory as giving us a *complete* definition of the concept of set, i.e., sets are *defined* as those things that together satisfy these particular axioms. The problem with this is that there are many competing axiom systems for set theory, and people really do disagree over which is the *true* theory, which raises the question arises as to whether "set" is really a well-defined concept at all. To put it another way, can we find axioms for a concept that captures the *essence* of set-hood while avoiding controversial details?

I believe that a good candidate for this concept is *extensionality*, which says that the *identity* of a set is completely determined by its membership; in symbols:

The Axiom of Extensionality: $\forall x(x\in s \Leftrightarrow x\in s') \Rightarrow s=s'$

Let's call the axiom system which has this as its sole axiom *membership theory*.

We now have two very simple axiom systems, *equality theory* and *membership theory*, which define two very basic concepts. How do these two systems compare?

Membership theory, though it does nothing beyond formalizing the concept of extensionality, is a first step towards a universal language for mathematics, which results from adding further axioms that refine the concept of *set*. Can the same be said of equality theory? As Gödel showed, the answer is most emphatically no! No matter what other axioms about the equality predicate we add to the equality axioms, the result will not be an even faintly interesting mathematical system, much less a universal system for mathematics. [3]

In brief, membership theory is an *open ended* axiom system, by which I'll mean an axiom system that can be made universal by adding more axioms. To put it another way, all of mathematics can be expressed in terms of the concept of membership. Equality theory, on the other hand, is a *dead end* axiom system, by which I'll mean an axiom system that cannot be made universal by adding more axioms. In fact, almost none of mathematics can be expressed in terms of the concept of equality alone.

A *universal system* is an axiom system with the property that any other consistent axiom system can be interpreted within it. [4] ZF set theory is universal, from which it follows that a necessary and sufficient condition for an axiom system A to be open-ended is that we can define a predicate $x \in y$ in A such that the ZF axioms expressed in terms of $x \in y$ are consistent with the axioms of A.

A *concept defining* axiom system is open-ended if all of mathematics can be expressed in term of its concept. Open and dead-end axiom systems correspond to open and dead-end concepts. What makes a concept dead-end is that in some sense it closes off the conceptual field to which it belongs. One way to open up a dead-end concept is to take away some of its defining axioms. For instance, if we take away Axiom 3 from equality theory the resulting weaker axiom system turns out to be open-ended [5] (of course if we take away all of its axioms, it becomes very open-ended!) At the opposite extreme are the so-called *complete* axiom systems in which every assertion is either provable or disprovable. Membership theory can be made complete, for instance, by adding the axiom that no set has more than one member, i.e. $\forall x \sim \exists y,z(y \neq z \ \& \ y \in x \ \& \ z \in x)$, and equality theory can be made complete by adding the axiom that everything is equal to everything else, i.e. $\forall x,y(x=y)$. There are also complete versions of arithmetic and set theory, but Gödel's incompleteness theorem shows that no complete axiom system can be universal.

Section 2. Identity

As we saw above, equality is a dead-ender, but what about *identity*? We have so far only encountered identity as a certain kind of equality, which would seem to put and end to any hope for its openness. But let me for the moment dispense with formalities and propose a more intuitive definition of identity that puts it in a rather different light:

Identity is that way of being the same that *matters*.

"But wait a minute" you complain "I thought we were doing mathematics and logic. What could it possibly mean in mathematics or logic for something to *matter?*"

We can pass on that question for the moment since all we need to consider here about mattering is that it doesn't happen all by itself. When something matters, it matters *to you*, or *to me* or, more generally, *to x*. The important thing to keep in mind is that identity, seen in the context of mattering, is a *three place relationship* among x, y and z which asserts that y and z are the same in the way that matters to x.

It's true that the word "identical" is sometimes used to mean the same in *every* respect, in which case the third term disappears and identity does become a special kind of equality. However, it's more common in everyday life to use "identity" in the sense defined above. For instance, suppose you hand your bank clerk a withdrawal slip and he asks to see your *identity* card, which shows that you are the *same* person who opened the account you are drawing on. Of course you are not *exactly* that same person; for one thing, you are five years older, newly married, and perhaps a few pounds heavier. Nevertheless, your card does establish that you still have the same *identity as a customer*, and that is what *matters* to the bank. We are now dealing with a three-place identity relationship in which the three terms are *you now*, *you then* and *the bank*.

The basic premise of this paper is that identity should be treated as a *three-place* predicate which asserts the identity of y and z *relative* to x. We could notate this in standard predicate form as, for instance, ID(x,y,z), but will better suit our purposes to give it a special notation:

The three-place identity predicate: x(y=z), read x regards y as the same as z.

What, then, are the axioms that define this new predicate?

If we replace x by the name of a possible value of x, for instance MutualBank(y=z), we transform the identity predicate into a *two-place* equality predicate, which of course must then satisfy the equality axioms. From this it follows that the axioms that define the concept of three-place identity are simply the equality axioms with the value of x left open. That is, whatever x may be, and whatever y, z and w may be, it is true that:

> Identity axiom 1) x(y=y)
> Identity axiom 2) if x(y=z) then x(z=y)
> Identity axiom 3) if x(y=z) and x(z=w) then x(y=w)

This is the requisite minimal set of axioms. But suppose we add more axioms? Might this create a universal axiom system, or is identity theory just another dead-ender?

Since the axioms for identity look so much like those for equality, there appears to be little hope for its openness. But appearances can be deceiving, and in fact it turns out that, far from being a dead-ender, identity theory is so open-ended that with only a few twists it can be turned it into *ZF set theory*!

We'll now perform this feat.

Section 3. The Universality Theorem

Universality Theorem: Identity theory is open, i.e. all of mathematics can be stated in the language of three-place identity.

What we'll actually prove here is that we can consistently place the axioms of ZF set theory [6] on a predicate $x\in'y$ that is defined in terms of $x(y=z)$. Since ZF is universal, this shows that identity theory can be made universal and is thus an open system.

Here is the proof:

Our first step is to define a new symbol $x\in'y$ in identity theory,[7] without any assumption as to what it means apart from this definition (I'll use the symbol $\in'$ rather than $\in$ for *defined membership,* with $\in$ reserved for membership in stand-alone set theories.)

D1. Membership defined in terms of identity: Let $y\in'x$ mean that x regards y as different from itself; in symbols: $y\in'x \;\Leftrightarrow:\; x(y\neq x)$. [8]

We of course cannot derive set theory from **D1** alone. The big question is whether we can *assume* the ZF axioms as statements about $x\in'y$ without contradicting the identity axioms on $x(y=z)$. If we can, then we will have *extended* the identity axioms in a way that makes identity theory universal, thereby proving the universality theorem. Of course we are assuming that the ZF axioms are consistent even though, because of Gödel's theorem, there is no way of proving that they are.

Proving consistency can be hard work, but in this case it turns out to be surprisingly easy. The first step of this proof is to define a new predicate $x(y=z)$ in ZF set theory that satisfies the identity axioms, as follows:

D2. Identity defined in ZF: Let $x(y=z)$ mean that both y and z are members of x, or both are non-members of x; in symbols: $x(y=z) \;\Leftrightarrow:\; (y\in x \;\&\; z\in x)$ OR $(\sim y\in x \;\&\; \sim z\in x)$.

We'll next apply **D1** to this defined $x(y=z)$ to define a predicate $x\in'y$ which will turn out to be *logically equivalent* to $x\in y$. That is, it follows from **D1** and **D2** that $x\in'y \Leftrightarrow x\in y$. This will show that $x\in'y$ also satisfies the ZF axioms, thereby proving that the ZF axioms are consistent with the identity axioms, which in turn proves the universality theorem.

But before we can take this step we must of course make sure that our defined $x(y=z)$ is actually an identity predicate, i.e. that it satisfies the identity axioms. To prove Axiom 1, $x(y=y)$, we replace z by y in **D2**, which turns the ZF sentence in **D2** that defines identity into $(y\in x \;\&\; y\in x)$ OR $(\sim y\in x \;\&\; \sim y\in x)$. Since this reduces to $y\in x$ OR $\sim y\in x$, which is logically true, we conclude that Axiom 1 is true. Note that this argument makes no use of the axioms of set theory, which shows that the truth of Axiom 1 is guaranteed by the logical form of D1 alone, i.e., it is true no matter what is meant by $x\in y$. The same holds for Axioms 2 and 3, though the proofs are somewhat longer. [9]

Our third step is to define a new predicate $x\in'y$ by applying **D1** to the identity predicate $x(y=z)$ defined by **D2.** Since $x(y=z)$ is defined in term of $x\in y$, then so is $x\in'y$. Since, by **D1,** $y\in'x \Leftrightarrow \sim x(y=x)$, substituting x for z in **D2** tells us that $x(y=x)$ means $((y\in x \;\&\; z\in x)$ OR $(\sim y\in x \;\&\; \sim z\in x))$. It follows that $x\in'y$ can be defined in terms of $x\in y$:

D3. $x\in'y \Leftrightarrow: \sim((y\in x \;\&\; x\in x)$ OR $(\sim y\in x \;\&\; \sim x\in x))$.

Finally, we must make use of a theorem of set theory, which is that no set is a member of itself, i.e., $\sim x\in x$.[10] Let F stand for *falsehood* and T stand for *truth*. We can replace the false statement $x\in x$ in **D3** by F and the true statement $\sim x\in x$ by T without changing the truth value of **D3**. The rules of Boolean logic then lead to the following series of logically equivalent statements:

$x\in'y \Leftrightarrow \sim((y\in x \;\&\; x\in x)$ OR $(\sim y\in x \;\&\; \sim x\in x))$.
$x\in'y \Leftrightarrow \sim((y\in x \;\&\; F)$ OR $(\sim y\in x \;\&\; T))$.
$x\in'y \Leftrightarrow \sim((F$ OR $(\sim y\in x \;\&\; T))$.
$x\in'y \Leftrightarrow \sim(\sim y\in x \;\&\; T)$
$x\in'y \Leftrightarrow \sim \sim y\in x$
$x\in'y \Leftrightarrow y\in x$

As mentioned, the logical equivalence of $x\in y$ and $x\in'y$ shows that $x\in'y$ also satisfies the ZF axioms, thereby proving that they are consistent with the identity axioms, which in turn proves the universality theorem.

Though the above proof is straightforward, I still find it rather mysterious that we can make the equality predicate open-ended just by relativizing it. There is a theorem in Section 3 of M&I called the *stereo equality theorem* that does shed a bit of light on this mystery: Consider a variation on equality theory in which there are *two* predicates satisfying the equality axioms and no other axioms. It turns out that we can also define a two-place predicate $x\in'y$ in this system on which we can consistently place the ZF axioms, thereby revealing the fact that two equalities taken together can express all of mathematics. Thus to open up the concept of equality we don't need to make it variable, just multiple.

Section 4. A theory of everything?

So what is the significance of all this? Is identity theory only a technical detail in the logic of axiomatics, or might it open up an entirely new way of thinking about the world? My intuition says “New thinking! Go for it!” The quiet voice of reason is more reserved: “If you really must, as you say, ‘go for it’, proceed with caution and beware the pitfalls of premature enthusiasm.” I’ll wind up this paper by tossing this well-meant advice to the wind and letting my intuition have its uncensored moment on stage.

Roger Penrose will be my first curtain-raiser. In his remarkable book about everything, “The Road to Reality”,[11] he begins by dividing reality into three worlds. First there is mathematical world, which he understands in Plato’s sense as an eternal world of ideal objects. Then there is the physical world, the world of matter in motion, to which certain mathematically described laws apply with astonishing precision. And then there is the mental world, which is mysteriously “evoked”, to use his term, by the physical world, and which closes the cycle with its mysterious ability to *comprehend* the mathematical world. [12] Later on in the book he hints that there may be a deeper level of reality underlying these three worlds: “We must bear in mind that each ‘world’ possesses its own distinctive kind of existence, different from that of the other two. Nevertheless, I do not think that, ultimately, we shall be able to consider any of these worlds properly, in isolation from the others.” [13] To this last sentence I say *amen*, and I’ll take it as my point of departure.

The mathematician-philosopher Gian-Carlo Rota will be my second curtain raiser: [14]

> “Speaking exoterically, the permanence of *identity* through a variety of possible or actual presentifications is the constitutive property of every item.”
>
> "Speaking esoterically, all physical, ideal or psychological presentifications of any item are secondary to the one primordial phenomenon which is called ‘identity’. Exoterically, the world is made of objects, ideas, or whatever; esoterically, it is made of items sharing one property: their permanence through each presentification. Identity is the ‘undefined term,’ and the properties of identity are the axioms from which we ‘derive’ the world.” [15]

If you found that heavy going, here is a livelier quote from the same essay:

> “Our exoteric slogan will be ‘Identity precedes existence’. Esoterically, the problem of existence is a *folie*”

Right on, Rota!

Is *identity* Penrose’s deeper level of reality? Might the three-place identity predicate provide the conceptual basis for a theory of *everything*? Before going down that path, let me introduce an alternative trinity to Penrose’s trinity of matter, mind and math that is not a trinity of worlds but of “whats”:

What exists
What matters
What happens

The word *exists* derives from the Latin *existere*, which means to stand out, and I'll use it in that sense here. For something to *exist* for x, i.e. for it to stand out to x, it must *matter* to x. It may *happen* that what exists for x today will no longer exist for x tomorrow. Note the cyclic relationship among the three "whats".

Let's first focus on what exists. An important issue is whether the x in x(y=z) exists for *itself*. Does x *stand out* to itself? Certainly x *matters* to itself. But mattering doesn't guarantee existence – if wishes were horses then beggars would ride. But the question remains: d*oes* x exist for itself? Concerning this, I can't resist a quote from William James, arguably the best of introspective psychologists:

"Introspection is like trying to turn on the light fast enough to see the darkness." [16]

As a Buddhist might put it, the *real* self is *non-existence,* but this is a delicate point. Let's simply assume that x, the "real" x, does *not exist* for itself.

If x regards itself as non-existent and also regards y as the same as itself, it must of course regard y as non-existent too. Is the converse true? If y is non-existent to x does it follow that x regards y as the same as itself? Let's assume that it does. Taken together with our first assumption, this leads to a formal *definition* of existence:

Existence defined: To say that y *exists* for x means that x regards y as *different* from itself; in symbols x(y≠x). In mystical terms, when I say that y exists I mean that I distinguish y from non-existence.

Recall that in our proof of the universality theorem, we defined y∈'x to mean x(y≠x), which we are now calling relative existence. In the context of set theory it would be stretching it to say that it *matters* to a set x whether or not y is among its members. But when set theory is interpreted within identity theory, membership is not actually equated with relative existence but is merely *encoded* as relative existence, which should allow us in good conscience use ∈' in identity theory as an *abbreviation* for relative existence. [17]

Existence notated: y∈'x means y exists for x.

I'm not sure what Rota meant by "the problem of existence" but perhaps he was referring to the question "What exists?" which has long been a favorite among philosophers. Kant shocked the world by announcing that material things don't exist, at least in our present sense of the word. That is, we can only be aware of the *appearances* of material things, not of the things *in themselves*. This doctrine, called *transcendental realism*, is summarized by the Cambridge Dictionary of Philosophy as asserting that "all of our theoretical knowledge is restricted to the systemization of what are mere spatiotemporal appearances." [18] If we omit the qualification "what are mere spatiotemporal", this is not so far from what I am proposing here. That is, it makes sense to paraphrase "y *exists for* x" as "y *appears to* x," or "y is *present* to x." If z also appears to x we can read x(y≠z) as saying that y and z are different *appearances* to x. What happens, then to physical objects? We'll deal with this problem in detail in M&I so let it suffice for now to say that *objectivity* will be taken to mean invariance under change of viewpoint.

On that note, let's move on to *what matters*. Our informal reading of $x(y=z)$ was that x regards y as the same as z in the way that matters. The negation $x(y\neq z)$ of $x(y=z)$ means that x *distinguishes* y from z in the way that matters. If z is nonexistent and $x(y\neq z)$ then y exists, so the existence of y matters. We'll see in M&I that when we look at complex compounds of three-place identity, what matters can take many forms. For now, though, I'll rest content with the observation that relative identity theory brings the *subject to whom things matter* into the core of formal reasoning. As I see it, one big challenge for identity theory is to use the subject-object polarity implicit in the contrast of outer and inner variables in $x(y=z)$ to explain the connection between *mattering* and *matter*.

What *happens*? Most people would say that for something to happen there must first be *time*. But this has it backwards. Time is the *chained succession* of events, but there are also a-temporal happenings, most notably *information* and *connectivity*. In M&I, these two concepts are shown to be the key to applying identity theory to physics. [19]

If we buy Rota's rather cryptic statement that "The permanence of *identity* through a variety of possible or actual presentifications is the constitutive property of every item," then it would seem that identity theory has its finger in every pie. So, is identity theory really the theory of everything?

Among physicists a theory of everything has come to mean a mathematical formalism that unites relativity and quantum mechanics in a single theory. I think that identity theory may be of some help here, a point on which I'll say more in M&I. But this unification is still a far cry from a theory of everything. Not only does it fail to take account of exotic new realms of being of which we still have no inkling, it even falls radically short of ordinary common sense, since it has nothing to say about *subjective* experience.

Consider the question "Why did you turn up the thermostat?" This is an ordinary commonsensical question to which I have an ordinary answer: "I was cold". This is an empirical fact that provides a perfectly good explanation. However, it's a *first person* truth, which in science is anathema. Similarly, science has no place for second or third person truths, so it cannot explain why *you* or *he* or *she* turned up the thermostat either.

For me, a theory of everything must encompass first, second and third person truths, which is not the same as dealing with *mental objects* like minds or egos or ids or thoughts or consciousness or whatnot. Though these can be helpful in relating inner truths to external facts, what I am asking for is a science that can deal *directly* with inner truths. This is no more than what we expect from a story teller or a historian who advances her narrative by putting us into the shoes of her characters, which reveals how far science has strayed from culture at large.

Here is where identity theory comes in. The statement "x(y=z)" is in effect a *third-person* statement, and indeed it's very natural to replace "x" by "he" or "she". When we do so, we implicitly bring in the first person, In order to understand "He is cold" I must remember what it's like for me to feel cold and then put myself in *his* shoes. Similarly, in order to understand "x regards y as the same as z" I must understand what it means for me to think "y and z are the same" and then put myself in x's shoes.

Let's recall Rota's pronouncement "Exoterically, the world is made of objects, ideas, or whatever; esoterically, it is made of items sharing one property: their identity [20] through each presentification." I, you, he and she all share the "property" of identity along with stones, words and numbers. It's true that in ordinary conversation, you, I, he and she are equally welcome in the company of stones, words, numbers, whatever. The "property" they share is the subject matter of identity theory. Why, then, should identity theory not become their abstract homeland?

To pursue these ruminations further requires a further technical development of identity theory, for which I'll refer the reader to M&I. To wind things up here, let me briefly return to the distinction we started out with, which is that between an "axiomatic" axiom system and a concept-defining axiom system. Which kind of axiom system is identity theory? In which way will it grow?

The choice here is between adding more *axioms* to encompass more *truth,* or else adding more *definitions* to encompass more *meaning*. In M&I we'll add one more axiom to identity theory, but after that it's definitions all the way up.

It's surprising how much new mathematics can be created just by adding new definitions. For instance, in M&I we'll define a single-place predicate ZF(x) in identity theory that creates ZF set theory as a *concept*, without adding any new identity axioms. We'll also define ZF_2(x) and ZF_3(x) that also create ZF set theory in identity theory, but in entirely different ways, and ZF_4(x) that expands ZF set theory into a *non-extensional* set theory. Definitions that define mathematical structures can be combined into richer structures. There is one example of this that is immediately relevant to quantum physics, which begins by creating the structure of Boolean algebra with a simple definition Boo(x), and then combines a range of such x's into a non-Boolean algebraic structure invented by von Neumann that he called *quantum logic*, whose structure is a representation of the abstract mathematical core of quantum mechanics.

All this is very well, but what I really find exciting about these structural definitions is the hope they raise for a conceptual home in science and mathematics for *you* and *I*.

REFERENCES

[1] Ref. [Note: Unfortunately this paper has not been completed. -- R. Shoup]
[2] Quine Philosophy of Logic Ch ?. … Give the whole spiel, starting with substitutivity.
[3] What Gödel actually proved is that the predicate calculus plus identity is a complete axiom system, from which it follows that there is no way of adding more axioms to it so as to make it universal. That nothing interesting can be created by adding more axioms is my own intuition, though I am ready to stand corrected.
[4] In Section 2 of M&I we'll formally define the concept of *interpretation*, which encompasses the concept of *model* but also allows for other ways of translating one axiom system into another.
[5] Here are the key steps in the proof; the full proof can be found in M&I

Define [x=y] to be equality without the third axiom, i.e. it need only satisfy $x=x$ and $x=y \Rightarrow y=x$.

Define [x =: y,z…] to mean that x equals y and z etc. and anything unequal to y is unequal to x, ditto z, etc. In symbols: ([x=y] & [x=z] & …) & (([y'≠y] ⇒ x≠y']) & ([z'≠z] ⇒ [x≠z']) & …)

Define $x \in' y$ to mean $\exists$x',x'',y',y'',q, q', q'' such that
[q =: x,q'] & [q' =: y,q,q"] & [x'=:x,x"] & [x" =: x,x'] & [y'=:y,y"] & [y" =: y,y']

There is as set model of [x=y] such that $x \in' y$ is membership in a set model of set theory. This theorem is at the heart of the proof of the so-called *stereo equality theorem* which we'll encounter later.

[6] We are of course assuming here that the ZF axioms themselves are consistent. Though Godel's theorem shows that this can't be proved, we have no reason to think that they are not.
[7] Remember that the identity predicate of set theory can be defined in terms of $x \in y$.
[8] The symbol ⇔: will only be used in the *definition* of a new predicate. The same sentence without the colon is an assertion of the logical equivalence of a simple sentence in the defined predicate to the sentence that defines it, and as such is not a definition but a *theorem* that follows from a definition. The notation $x(y \neq z)$ of course means $\sim x(y=z)$.

[9] The second axiom,, $x(y=z) \Rightarrow x(z=y)$, translates into
$((y \in x \ \& \ z \in x) \vee (\sim y \in x \ \& \sim z \in x)) \Rightarrow ((z \in x \ \& \ y \in x) \vee (\sim z \in x \ \& \sim y \in x))$
which is a logical truth of the form $A \& B \Rightarrow B \& A$.

The third axiom, $x(y=z) \ \& \ x(z=w) \Rightarrow x(y=w)$, translates into

$((y \in x \ \& \ z \in x) \vee (\sim y \in x \ \& \sim z \in x)) \ \& \ ((z \in x \ \& \ w \in x) \vee (\sim z \in x \ \& \sim w \in x)) \Rightarrow ((y \in x \ \& \ w \in x) \vee (\sim y \in x \ \& \sim w \in x))$

To more easily see its logical form, let's abbreviate it by letting *a* mean $y \in x$, *b* mean $z \in x$, and *c* mean $w \in x$, and also abbreviating conjunction and disjunction, i.e. we'll write a&b as *ab*, and OR as +. This shortens it to: $(ab + \sim a \sim b)(bc + \sim bc) \Rightarrow (ac + \sim a \sim c)$, which, after distributing terms in the premise, becomes

$(abbc + ab\sim b\sim c + \sim a\sim bbc + \sim a\sim b\sim b\sim c) \Rightarrow (ac + \sim a\sim c)$.

We can drop the middle terms of the premise because they are false. After eliminating redundancies, we end up with $(ac + \sim a\sim c). \Rightarrow (ac + \sim a\sim c)$, which no one would argue with.

[10] This is not only true in ZF but in any of its serious competitors.
[11] ..ref.
[12] Chapter 1, Section 4, pp 17- 21 Penrose's trinity is what philosophers would call an *ontology*, which, roughly speaking, is a *classification of everything*. Ontologies have various degrees of refinement, and

differ greatly in what they consider to be the fundamental categories, but they have in common their determination to leave nothing out. But, and here is the rub, if you don't want to mistakenly leave something out you must be able to *identify* that thing, either alone or collectively. In short, you must be committed to an *absolute* concept of identity.

[13] Chapter 34, Section 4, p 1030

[14] This quote is from the essay *The primacy of identity* in the book *Indiscrete Thoughts* by Gian-Carlo Rota, Birkhauser Boston 1997

[15] Rota does not list the so-called "properties" of identity, though he esoterically identifies them with *axioms*. As far as I know, though, he had not meant this in a formal sense, and was not thinking of the axioms of of relative identity as the axioms I have presented above, though I have reason to believe he would have been sympathetic to this idea. *Identity* in either its traditional two-place or its three-place form is at home in all three of Penrose's worlds of "things", "thoughts" and "math", which roughly correspond to Rota's "objects", "ideas" and 'whatever". However, traditional identity forces us to make an absolute separation between the *objects* of these three worlds: thoughts are *not* physical object, physical objects are *not* mathematical objects, and mathematical objects, though they are the *objects* of thoughts, are not themselves thoughts. Three-place identity puts us under no such compulsion.

[16] … ref James' *Psychology* etc.

[17] This definition of relative existence does not imply extensionality for $\in$' so it would be misleading to call $\in$' membership. Though we know from our proof of the universality theorem that the extensionality of $\in$' is consistent with the identity axioms, it would greatly diminish the expressive power of identity theory to add it to the identity axioms.

[18] … ref

[19] The distinction between time and change, far from being a wild new idea, is of the essence in Shannon's concept of *information*. To *gain* information means to *narrow* the range of what is possible. To *lose* information is to *widen* that range. Information itself is that which is common to the *gaining* and *losing* of information. When we say that a disk *has* 5 GB of information we are not contemplating anything that is actually happening; rather, we are contemplating 5 GB of instances of that which is common to the doubling or halving of what is possible.

A *connection* can be understood as an item of information in which the wider range of possibilities, called the *open connection*, is the set of possible joint values for a certain pair of variables, and the narrower range, called the *closed connection*, is that subset of the wider range for which the members of the pairs are equal. Example: Open connection: wave you arms around. Closed connection: wave your arms around with your hands clasped.

To *factor* a connection into *partial connections* means to identify the two variables of its *open* state with *component* variables, for instance, to identify the two spatial variables of one's two hands with their x, y and z coordinates, thereby factoring the handclasp connection into an x-connection, a y-connection and a z-connection. There are in general many different ways to factor a connection. Some of these have been shown to lead to the core laws of quantum mechanics. [19] Others, in which we take into account the arbitrariness in choosing among the ways of factoring, may lead to new principles of relativity, and this is a new hope for bringing space-time into the quantum core.

These results can be formally stated in the language of set theory via the theory of relations, where a *relation* is defined as a set of ordered n-tuples. There is one rather odd thing we must do to make this to work for quantum mechanics, which is to divide a set of n-tuples into *negatives* and *positives* which cancel when we count cases. The resulting so-called *link theory* of quantum mechanics is mathematically equivalent to the standard wave-mechanical formalization, but it places quantum processes in the company of a much broader class that encompasses both classical Markov processes and a whole range of other processes that we have not as yet looked for in nature. My intuition is that we will not even be in the ballpark of a theory of everything until we allow ourselves to look in that direction. But even supposing we do, we are still missing something crucial, which is *ourselves*.

[20] I've substituted "identity" for his word "permanence", which is too specialized.