For Jim Bowery

Membership and Identity

File name: **Membership and Identity 9-13-06** Updated: 9-15-06

Sec2 p5, Sec3 p12, Sec4 p18

Alt-n symbols: **1**∈ **2**⇔ **3**⇒ **4**≡ **5**≠ **6**∩ **7**∪ **8**⊂

By Tom Etter

"Identity precedes existence" Gian-Carlo Rota

Section 1. Zermelo's set theory

An ontology, or more exactly, a *working* ontology, is a classification of those things that we think about and work with. It's true that when philosophers speak of ontology they often have in mind a more ambitious project, which is the classification of *everything*. However, it's not hard to expand a humble working ontology into one that classifies everything: just add one more category called "Everything else", or "Other" for short. In Section 3 of this paper we'll see that this is more than just a verbal trick and is actually a very useful step, somewhat like making zero into a number. For now, though, we'll stick with working ontologies in the narrow sense.

The naive working ontology of mathematics has categories like numbers, functions, shapes, spaces, groups, etc., each with many sub-categories. In the nineteenth century, as mathematics became more abstract and complex, efforts were made to transform this sprawling informal ontology into a more orderly one based on more fundamental categories. The culmination of these efforts was modern set theory. But before we come to set theory proper, let's take a brief look at an important early step towards this goal of unification that occurred around 1870 when Dedekind discovered a new and surprising way to represent irrational numbers as *sets* of rational numbers.

The problem of "irrational" magnitudes such as the square root of two had been around since antiquity. Pythagoras solved it to his own satisfaction by flatly denying that there are such things.[1] His solution didn't acquire much of a following, though; after all, the diagonal of the unit square does have a *length*, and what is a length if not a number? So *irrational* numbers have remained with us ever since – they were called irrational because they didn't make sense in terms of counting. The unfortunate geometer, who couldn't do without them, had no choice but to accept them as visitors from an alien ontological category.

Enter Dedekind. What is the square root of two? It's simply the *set* of all ratios whose squares are less than two. Not an alien in sight. More generally, an irrational number is the lower part of a division of the set of all rational numbers into two parts, where this lower part has no last member and the upper part has no first member. [2]

Reason does rebel a bit at this. How can a set *be* a number? Sets *have* numbers, which are called their *cardinalities*, but they are not *themselves* numbers, and even if they were, the number of things in the above mentioned Dedekind set has no relationship at all to the square root of two, being in fact infinite. Dedekind won the day, however. Just stick with my definition, he insisted, and I'll tell you how to add, multiply etc. so that all your calculations come out right. And so we did, and so they did.

Dedekind's conceptual invention was one of the first instances of sets being treated as mathematical objects in their own right, and, because it solved an ancient puzzle, it became a key step in turning set theory into an accepted branch of mathematics. Though the *logic* of sets had been around since Aristotle, logic was generally regarded as a different enterprise from mathematics, even after Boole formulated his algebra of sets. By 1880, however, that situation had changed completely, and Cantor, the inventor of infinite numbers, went so far as to say that set theory *is* mathematics, the *whole* of mathematics! Was Cantor right?

Certainly many people think so today. Set theory did have a bit of a setback when the so-called new math for children, which was based on set theory, turned out to be a fiasco. However, if you open almost any advanced mathematics book for adults you'll find that everything it says is implicitly grounded in set theory. The reigning ontology of mathematics has only one basic category: *sets*. Every mathematical object is a set. Numbers are sets, functions are sets, spaces are sets, groups are sets, etc. etc. etc.

"But what about the members of these sets?" you ask. "It's true that you can take sets of sets of sets etc., but if you keep going in the other direction, mustn't you eventually arrive at things that *aren't* sets?"

Actually, no. Zermelo, in around 1900, found a way to get completely rid of non-sets by *founding* all sets on the so-called *null set*, the set without members. We start out with the null set, written "{}", and form the set {{}} of which {} is the sole member, then the set {{{}}} of which {{}} is the sole member, then the set {{{}},{}} which has the sets {{}} and {} as members, then on to {{{{}}}, {{}}, {}} etc. etc.

The most fundamental axiom of Zermelo's and every other set theory is the so-called *axiom of extensionality*, which says that the identity of a set is entirely determined by its membership, i.e. if x and y have the same members then they are the same set. Indeed, this feature of the membership predicate $x \in y$ is basically what *defines* set theory as such. To ensure an adequate supply of sets, Zermelo posited several axioms of *closure*, among them the unit-set axiom which says that for any set there is another set of which it is the sole member, the pair-set axiom which says that for any two sets there is a third of which they are the two members, etc. He also added a very strong assumption in the form of a *meta*-axiom [3] that involves both sets and the language of sets:

The meta-axiom of separation: Given any property P that can be expressed in set language, and any set s, there is a subset p of s consisting of all members of s that have the property P.[4]

Older set theories had posited that for any property P there exists a set of *all* things having P. However, this stronger assumption was shot down by Russell's famous paradox, which arises when we let P be the property of not being a member of itself. Is the corresponding set a member of itself? If it is then it is not, but if it is not, it is.

Though there have been, and still are, many versions of set theory, Zermelo's is the one that is generally favored today, at least by mathematicians.[5] Surely one reason is that it convincingly avoids Russell's paradox. But another is the simplicity and power of its ontology. To get a feeling for why some people regard set theory as *the ontology* of mathematics, let's see how it defines whole numbers as sets. But first, a bit of history.

The original set-theoretic definition of (whole) numbers came from Cantor and Frege in the late 19th century. It begins by observing that for two sets to have the *same number* of members means that there exists a *one-one correspondence* between the members of one and the members of the other. Cantor called such sets *equivalent.* We'll keep this useful term despite its possible ambiguity, and we'll write $x \equiv y$ to mean that x and y are Cantor-equivalent.

All this is fine as far as it goes, but it raises the question that if *everything* is a set, then what kind of a set is a one-one correspondence? The standard answer is that it's *a set of ordered pairs* that uniquely matches the members of one set with the members of another. But, then, what kind of a set is an ordered pair? The standard answer is that the ordered pair $\langle x,y \rangle$ of x followed by y is the set $\{\{x\}, \{x,y\}\}$. But why that particular set? Well, like Dedekind's irrational set, it does the job. But is this really a good enough answer? Wouldn't the job be done just as well if we defined x followed by y as, for instance, the set $\{\{x\}, \{\{y\},y\}\}$? After all, there is nothing mysterious or irrational about the concept of x followed by y, so why do we have to encode it as such an arbitrary structure?

We'll return to questions like these, but we still have a way to go in defining numbers themselves, since all we have done so far is define *same number*. Cantor and Frege thought they had what is obviously the right answer, which is that a (whole) number is an *equivalence class* of sets under Cantor equivalence, i.e. it is the set of all sets that are equivalent to some set S.[6] But this definition, since it involves *all* sets of a certain kind, was again rudely shot down by Russell's paradox, and though Russell and others made determined attempts to save it in a more complicated form,[7] it has little appeal today.

What we find in modern textbooks on set theory is a very different kind of definition. The number n is no longer defined as the set of all n-member sets, but as a certain *exemplary* n-member set. These exemplary sets are constructed as follows: We start with 0, which is defined as the set with no members, i.e. the null set $\{\}$. The number 1 is then defined as the set $\{\{\}\}$ whose sole member is the null set. Next, the number 2 is defined as the set whose two members are 0 and 1, etc. etc. The general rule is that the number n is the exemplary n-member set $\{0, 1, 2, \ldots n\text{-}1\}$.

This scheme was originally devised by Dedekind for finite numbers, but von Neumann extended it to infinite numbers and also coined the term "counter" for sets of this type, which are now known as *von Neumann counters*. .

Again, as with ordered pairs, we ask "Is the set {0, 1} *really* the number 2?" And again the usual answer is that the scheme *works*, so what more do you want? Perhaps what more you want doesn't immediately spring to mind, but you are nevertheless feeling a bit dissatisfied with what you have got. As you progress further through a modern textbook on set theory, this dissatisfaction is not likely to abate. It's true that, in the appropriate context, the newly defined set entities do the same job as their naïve predecessors. But then an automobile, in the appropriate context, does the same job as a horse and buggy, but must we say that an automobile *is* a horse and buggy?

Is every mathematical object *really* a set? Was Cantor right?

There is indeed a great deal that is good and right about set theory. The question is whether we have to swallow its ontology whole in order to keep what is good and right about it. In the next section we'll explore a new way to think about the ontology of sets that does a better job of fitting the definitions of mathematical items as sets to the items themselves.

Section 2. The third way

The problems we have so far encountered have to do with the *identity* of mathematical items.

Identity is a slippery word.

"What is time?" asked Augustine in a famous passage in his Confessions. "When I don't ask, I know. When I ask, I don't know." For "time", substitute "identity".

So what is identity? When I don't ask, I know. When I ask, well, I do find I have a lot to say about the subject, but … OK, I don't know. Here goes, anyway:

One meaning of *identity* is found in its adjectival form *identical*, which means *the same in every way*. But, then, what does "the same" mean? Let's not ask, at least not yet.

A very different meaning emerges when I show my *identity card* to an official. Suddenly I become, in his eyes, that *person* to whom that card was issued many years ago, even though today I bear little resemblance to that person.

The identities of people, and of most things in life, survive certain kinds of change but not others. Consider the following conversation: "That's John, the man I saw yesterday." "But he's wearing a different hat." "Never mind, I'm sure he's the same man." The identity in question is John's identity *as* a *man*, not his identity *as* a *hat wearer*. The magic word here is "as".

Issues of identity can get murky. Suppose that John had had a sex change the previous night. Then John would not be the same *man*, though she would presumably still be the same *person*. But what if he-she had also had a brain transplant? Things get murkier. Fortunately we don't have to deal with this kind of murk here since we are working in the much simpler-minded world of mathematics.

We started this essay with irrational numbers and saw how Dedekind's way of defining them as sets of rational numbers gave a boost to the then nascent mathematics of sets. This happened well before the invention of modern set theory, when rational numbers were simply taken for granted as the members of an ancient mathematical category. But, after Cantor and Zermelo, rational numbers, like whole numbers and other mathematical objects, also became fair game for set theory.

We have already seen how Zermelo's set theory captures whole numbers as von Neumann counters. Rational numbers are represented in practice by *ratios* of whole numbers, which in grade school we called *fractions*. In set theory, the fraction m/n is represented by an ordered pair <m,n> of whole numbers which, as we saw above, is defined as the set {{m}, {m,n}}. Thus the fraction ½ is {{{{}}},{{{}},{{},{{{}}}} (try explaining that to a third-grader!). In grade school, when the teacher tells us that 2/4 =

½, no student raises his hand to complain that 2/4 and ½ are different *sets*. At that age we know very well that this doesn't *matter* because they are the same *as numbers*, and that's what *matters*. More generally:

Identity is that way of *being the same* that *matters*.

We won't ask what it means to *matter*, at least not yet.

Since a rational number can be represented by any one of an infinite number of different fractions, how do we capture it as a set? There is a Cantorian answer and, if you'll pardon the expression, a von Neumannian answer.

The Cantorian answer has the same general form as Cantor's definition of whole numbers. Recall that he started by defining sets as *equivalent* if they can be put into 1-1 correspondence, and then defined (whole) numbers as their equivalence classes, a definition that, as we saw, was shot down by Russell's paradox and replaced by von Neumann's definition of counters. But a definition of the same form as Cantor's does in fact work for rational numbers. Define fractions n/m and n'/m' to be *equivalent* (normally, but incorrectly, written n/m = n'/m') to mean that nm' = mn' (now we are using the equality predicate correctly, since we are taking whole numbers to be von Neumann counters, which are sets). Rational numbers are then defined as equivalence classes of fractions under this new form of equivalence. These equivalence classes, unlike Cantor's, don't lead to Russell's paradox, since they turn out to be quite unproblematic sets whose existence is guaranteed in Zermelo set theory by axioms that no one has yet found suspect.

The von Neumannian answer is simpler. Just as a von Neumannian whole number n is a certain *exemplary set* with n members, we can define a von Neumannian rational number r as a certain *exemplary fraction* that represents r, namely the fraction with the smallest numerator in its equivalence class. Thus ½ is the von Neumannian rational number among the equivalent fractions ½, 2/4, 3/6, etc. Note that, as in the case of von Neumann's counters, this exemplary fraction is uniquely simple and has no serious competitors. Such obviously superior exemplars are sometimes called *canonical.*

We've just seen that set theory can capture rational numbers in either of two ways, as sets of equivalent fractions, or as canonical fractions, where capturing means interpreting as sets. But there is also a *third way.*

This third way starts out just like the Cantorian way, by defining some notion of equivalence on a certain class of sets. But then the two ways part company. Instead of identifying our new objects as *classes* of equivalent things, the third way creates them *by fiat* by declaring that its new kind of equivalence is the *kind of sameness that matters*.

For instance, when we are dealing with rational numbers, we say that 2/4 = ½, by which we mean that 2/4 and 1/2 are the same *in the way that matters.* John today is wearing a brown hat, but he is the same *man* we saw yesterday wearing a white hat, and that is what *matters*. 2/4 is a certain number wearing a brown hat and ½ is that same number wearing a white hat, so to speak. The third way is a kind of deliberate return to our childhood innocence in using the word "is", but it's a return within the adult world of modern logic.

The third way really begins to show its power when we apply it to *whole* numbers. Cantor's definition of *same number* is obviously right; is has no competitors. The third way not only agrees that it's right but goes further to insist that it's *the whole story.*

"Wait a minute" you loudly complain "Never mind hats and mattering identities, just tell me how to *point* to these weird mathematical objects, or non-objects, that you would call numbers. What, according to your third way, are the numbers *themselves*?"

Good question, but let me slightly rephrase it: What, when we take the third way, do the names "1", "2", "3" etc. *refer to*? If they don't refer at all, what happens to arithmetic? How can we prove that 1+2 = 3 if we can't even *say* it?

Fortunately, Bertrand Russell, in his so-called theory of *definite descriptions*, invented a logical trick that comes to our rescue here. Even though we do take the third way we can still give numbers their usual names, and "1+2 = 3" still means what it says and is still provable. Actually, it comes closer to meaning what it says than when it is "encoded" as a statement about von Neumann's counters.

Here is how Russell's trick works:

First of all, names in mathematics are always introduced by definite noun phrases of the form "The thing x such that D(x)", where D is a property that applies to one and only one thing. D(x) is what Russell calls a *definite description*. For instance, if D is the property of having no members, which applies to only one set, then the definite description "The thing x such that D(x)" defines the name "{}". Sometimes we think of the whole noun phrase as having a reference, but, properly speaking, it's the pronoun "x" that does the referring, i.e. that is literally the *pointer*.

Though x does the pointing, it's the description that does the real work. To take another example, consider the sentence "The President is out to lunch", i.e. "The thing x that Presides is out to lunch." Another way to say this is "Whatever x may be, if x Presides then x is out-to-lunch". Notice that this second sentence contains no nouns or noun phrases (remember, "x" is a pronoun, and we can regard "out-to-lunch" as an adjective). This way of rephrasing noun phrases in terms of definite descriptions can be applied to any sentence to produce an equivalent noun-free sentence.

Suppose we give a name to the thing that Presides; let's call it Mr. B. We can than use this name to considerably shorten the noun-free sentence "Whatever x may be, if x Presides then x is out-to- lunch". We first step backwards to the equivalent sentence "The thing x that Presides is out-to-lunch". Since we have *defined* Mr. B as "the thing x that Presides",[8] our sentence now turns into "Mr. B is out-to-lunch". This is what Russell had in mind when he spoke of names as shorthand.

Russell's theory of definite descriptions can be summarized in four observations:

1) Names in mathematics are defined by definite noun phrases.

2) Definite noun phrases can be eliminated in favor of definite descriptions.[9]

3) This operation is reversible.

4) Definite noun phrases can be abbreviated by names.

These observations don't always apply in the everyday world where we can, like Adam, just point to things and say "apple", "tree" "woman" etc. However, in the simplified world of mathematics, they would appear to be unexceptionable. I very much doubt that it is possible to actually do mathematics without names, but, so long as we are willing to treat names as shorthand, we can grant Russell that they are unnecessary in principle.

So how does the theory of definite descriptions relate to the third way?

First of all, definite descriptions are inseparably bound up with *identity*. To definitely describe something is to *identify* that thing. For D to be a definite description, two requirements must be met: first, there must *exist* an x for which D is true, and second, this x must be *unique*, i.e., if D is true of both x and y then x and y are *identical*; in symbols, $\exists x(D(x) \; \& \; \forall y(D(y) \Rightarrow y=x))$.

When Russell invented his method of definite descriptions, the identity predicate "x=y" was assumed to be unique and absolute, as it still is today in most expositions of logic. But when identity becomes *relative*, as it does when we take the third way, then of course *definiteness* in Russell's sense becomes relative too, and this is indeed a major change. It dictates that before we can say that D is *definite* we must first say what we mean by *identity*. To name an object we must now provide both a description of that object and an identity predicate with respect to which that description is definite.

Is this the end of things-in-themselves? Are *objects* no longer *objective*? If so, what happens to *ontology*? We'll return to questions like these in the next section.

We still have some unfinished business concerning numbers. Though we managed to save the essence of Cantor's definition of number by using his equivalence predicate $x \equiv y$ as our identity predicate and then using Russell's trick to give names to those things that are identified by $x \equiv y$, the ontology of arithmetic in practice contains not only numbers but *sets* of numbers. This presents a problem, since in Zermelo set theory the members of a set are themselves *sets*, identified by set identity $x=y$, not by Cantorian identity $x \equiv y$. Of course this problem doesn't arise when we take numbers to be von Neumann counters, which *are* sets, but with Cantorian numbers it is inescapable.

One way to solve this problem is to define a new class of abstract entities called *number-sets* by introducing yet another new identity predicate on *representative sets*, where a representative set is defined by the condition that no two of its members have the same cardinality. This new identity says that representative sets x and y are identical if and only if there is a one-one correspondence between them such that corresponding members are Cantor-equivalent. For instance, a representative of the number-set whose members are 1, 2 and 4 is any set {x,y,z} in which x has one member, y has two members and z has four members. Any other set {x',y',z'} satisfying these conditions is also a representative and is thus identical to {x,y,z} *as* a number-set. Note the analogy to fractions as representatives of rational numbers, which we can push further by saying that a representative of a number-set is *canonical* if its members are von Neumann counters.

But is this really a good solution? These so-called number-sets can be made to do the same jobs as sets of numbers, but are they *really* sets? One thing is sure: they are *not* sets in Zermelo set theory. But haven't we then fallen back into the quagmire of arbitrary encoding that the third way was supposed to get us out of? There are depths here that go beyond the scope of the present paper, but let me briefly inject a passing subversive thought: Might it be that the concept of *number* is *complementary* to that of *set* in a way analogous to the complementarity of position and momentum? If we take the third way, can we only speak literally about one by ignoring or encoding the other?

Let me conclude this section with some reflections about identity and the third way that apply not only to set theory but to axiom systems in general.

Identity predicates belong to the wider class of predicates called *equality* predicates which are not defined by how they function in an axiom system but only by the formal axioms of reflexivity, symmetry and transitivity – in symbols:

Reflexivity: $x=x$
Symmetry: $x=y \Rightarrow y=x$
Transitivity: $(x=y \;\&\; y=z) \Rightarrow x=z$.

One of the most fundamental relationships among equality predicates is *refinement*. We say that one equality predicate *refines* another if it makes finer-grained distinctions. Thus set identity refines Cantor equivalence, i.e. $x=y$ implies $x\equiv y$ but $x\equiv y$ does not imply $x=y$. The converse of refinement is *partition*. Cantor equivalence *partitions* set identity, which itself of course refines any other defined equality in set theory.

The third way has led us to three kinds of objects abstracted from sets: rational numbers, Cantorian numbers, and number-sets. But can sets *themselves* be abstracted from more fundamental entities? Are sets the rock bottom? Does the third way only go *up*, not *down*? Or might it be that set theory can be abstracted from a theory whose identity predicate is more refined than set identity?

In Section 1 we noted that the defining axiom of set theory is the axiom of extensionality, which says that if x and y have the same members then x and y are the same set, no matter how differently they are defined. Thus the set of all odd numbers less than eight is set-identical to the set of all prime numbers less than eight, even though being *odd* is a very different criterion for membership than being *prime*.

There have in fact been many diverse proposals to re-define sets so as to take into account such finer-grained distinctions. Non-extensional sets are sometimes called *intensional sets*, though I think that a better word for them is *classes*, as in *classification*. To quote from the Cambridge Dictionary of Philosophy: "The word 'class' is sometimes used as a synonym for 'set'. When the two are distinguished, a class is understood as a collection in the logical sense, i.e. as the extension of a concept (e.g. the class of red objects)", it being implicit that the concept of red is a part of its identity. If we should make this explicit, then the resulting *class theory* would be a *refinement* of Zermelo's set theory, a step *down* on the third way. We'll come back to classes in the next section.

In conclusion, let us reflect on how the third way bears on the concept of a *model.* "Model" today is a buzzword whose meaning is often a bit vague, but it does have a precise technical meaning in logic, thanks to the logician Tarski. His definition is too specialized for our present purposes, though, [10] so I would like to propose another that is better suited to our needs and also, I believe, closer to popular usage.

First of all, I agree with Tarski that to model an axiom system *A* is to describe a relationship between *A* and an *exemplification* of what is formally asserted by *A*. But for Tarski, and here is where I have problems, this exemplification is a *mathematical object*, and an object of a specialized kind, namely a collection of ordered n-tuples. For instance, there is a Tarski model of set theory within set theory itself that results from interpreting $x \in y$ to mean that the ordered pair $<x,y>$ is in the collection of all ordered pairs $<x.y>$ in which x is a member of y.

It would be one thing if ordered n-tuples were physical objects like the pieces of balsa-wood and glue that we make model airplanes out of, but in Tarski modeling the pieces are *theoretical objects* that are usually defined within some other axiom system. Thus the essence of modeling is not a relationship between language and *things*, but a relationship between axiom systems, one of which is *interpreted* in another.

Interpretation. To *interpret* an axiom system *A* within an axiom system *H* means to identify the primitive predicates of *A* with defined predicates in *H* in such a way that the axioms of *A* become theorems of *H*. The axiom system *H* will be called the *host* of *A*.

What distinguishes models from other kinds of interpretations is that the objects in *A* are interpreted as objects in *H*. This is true in the above Tarski model of set theory, since the modeled predicate $x \in y$ applies to the two members of $<x,y>$, which are sets. It is also true of the *counter* interpretation of arithmetic, which, though it is not a Tarski model, should certainly qualify as some kind of model. It is not, however, true of our third-way interpretation of numbers, whose Cantorian identity predicate *partitions* the identity predicate of set theory. Here, in more general terms, is this distinction:

Model: an interpretation of one axiom system in another whose identity predicate matches [11] the identity predicate of its host.

The first and second ways in set theory produce models, but the third way produces abstract interpretations. When we interpret arithmetic by defining the arithmetic operators *plus* and *times* as set-theoretic operators on von Neumann counters, we are creating a model of arithmetic. But when we interpret them as three-place predicates whose *intrinsic* identity predicate is x≡y, we are *abstracting* arithmetic from set theory, not modeling it in set theory [12] By the *intrinsic identity* of a predicate P, I mean the least refined equality predicate that is *substitutive* in every place of P; the details of this definition will be spelled out in Section 4.

Though I have been arguing in favor of abstract interpretations and implicitly frowning on models, I must concede that models can be very useful tools, and indeed Section 4 is mostly about set-theoretic models of relative identity. In dealing with models the important thing is to not confuse the objects being modeled with the objects in the ontology of the modeling language. The set {{}} is *not* the number 1. What, then, *is* the number 1? It's what is abstracted from one-member sets by Cantor identity. But just how does this real 1 differ from {{}}? This raises a puzzling question, which is if {{}} and 1 are really *different*, then shouldn't there be an *identity* predicate that distinguishes between them? Such a predicate can't be defined in set theory, but we'll see in Section 3 that it *can* be defined in an axiomatic formulation of *relative* identity theory.

Section 3. Things, identity, and Other

I began Section 1 by defining an *ontology* as a *classification of things,* and a *working* ontology as a classification of things that *matter*. When our full attention is on what matters, what *doesn't* matter disappears into a black hole that goes under many names: absence, elsewhere, irrelevance, unimportance, nothingness, *Other*. An ontology that has an Other, that includes what doesn't matter as well as what does, will be called *complete*.

These are rough definitions, but *ontology* is an elusive concept. Another rough definition of ontology is a theory of *what exists*. For instance, the materialist believes that material things alone are what exist. But where does this leave immaterial things that *don't* exist, such as the things we encounter in dreams? Which raises the question of just what is a *thing*? That may sound like a silly question, but here is my answer anyway:

Thing (n., singular)**:** anything that can be distinguished from something else.

"Whether or not your question is silly, your answer certainly is, since as a definition it is blatantly circular" you complain.

Indeed it would appear to be circular,[13] but appearances can be deceiving. In fact it is not circular when properly translated into the language of mathematical logic. Here is a famous passage from Martin Buber's classic book *I and Thou* which, though not exactly about thing-hood, gives us a hint of how such a translation may be possible.

> The attitude of man is twofold, in accord with the twofold nature of the primary words which he speaks.
>
> The primary words are not isolated words, but combined words.
>
> The one primary word is the combination I-Thou.
>
> The other primary word is I-it.

Thus begins a deep inquiry into the human condition, but this is not the place to follow Buber in that direction. My reason for bringing up this passage is its surprising claim that *compound* words can be more fundamental than their *parts*. Buber insists that we don't start with an *I* and a *Thou* which then come together into an *I-Thou* relationship, nor with an *I* and an *it* which then come together into an *I-it* relationship, but that the *relationship* itself is primary and that the word "I" by itself is a broken-off fragment.[14]

Whatever the merits of Buber's claim, I bring it up here as a model for a claim of my own, which is that the word "thing" is a broken-off fragment of the more fundamental compound words "anything" and "something". That these words are fundamental is hardly debatable, since they are two of the four fundamental words of symbolic logic,[15] where they are written as $\forall$ and $\exists$. With this in mind, let's reexamine the above definition of a thing as *anything* that can be *distinguished* from *something* else.

For things to be *distinguished*, someone or something must *distinguish* them; let's call him, her or it the *discriminator*. Since discriminators vary both in the power and in the scope of their discriminations, discrimination is a *three-place* relationship:

The discrimination predicate: $x(y \neq z)$ means that x can distinguish y from z.

And then there is the three-place *indiscrimination* predicate, or *relative identity* for short,

The relative identity predicate: $x(y=z)$ means x can't distinguish y from z.

As we saw in the last section the *two-place* indiscrimination predicate, otherwise known as equality, is defined by the three equality axioms: *reflexivity*, *symmetry* and *transitivity*. Our formal definition of relative identity relativizes these axioms by adding to the equality predicate a third place for the discriminator:

Relative identity axiom 1: $x(y=y)$ [16]
Relative identity axiom 2: $x(y=z) \Rightarrow x(z=y)$)
Relative identity axiom 3: $x(y=z)$ & $x(z=w) \Rightarrow x(y=w)$

Since x is universally quantified, it follows from our axioms that *everything* is a discriminator. "But isn't that panpsychism?" you complain. Well, maybe, though I'm more inclined to think of it as a step towards a universal mathematics of relativity. However, we will take another look at panpsychism in Section 4.

Discrimination and relative identity are *contraries*, i.e. $x(x \neq y)$ means $\sim x(y=z)$, and $x(y=z)$ means $\sim x(y \neq z)$, so when we substitute $\sim x(y \neq z)$ for $x(y=z)$ in the relative identity axioms we get the defining axioms of the discrimination predicate:

Discrimination axiom 1: $\sim x(y \neq y)$
Discrimination axiom 2: $\sim x(y \neq z) \Rightarrow \sim x(z \neq y)$
Discrimination axiom 3: $\sim x(y \neq z)$ & $\sim x(z \neq w) \Rightarrow \sim x(y \neq w)$

Which is the *primary* concept? Followers of Hegel and Spenser-Brown will certainly opt for discrimination, but mathematicians and philosophers are usually more comfortable with identity. I find the two equally primary. However, I do find it easier to work with the relative identity axioms, which I think of as a simple stand-alone axiom system:

Relative identity theory: The axiom system based on the three relative identity axioms.

These three axioms define the *subject matter* of relative identity theory, but leave room for further axioms that say something more *about* that subject matter. The same is true of the axiom of extensionality, which defines the subject matter of set theory but leaves room for many different *kinds* of set theory. Similarly for the group axioms that define the subject matter of group theory, or Hausdorff's axioms that define the subject matter of topology. Let's call such axiom systems *topic-defining*. We usually add other axioms to topic-defining axiom systems to create richer and more useful systems, or what amounts to pretty much the same thing, we use them to classify structures within richer axioms systems, for instance topological groups in geometry.

While we're at it, let's add *equality theory* to our list of topic-defining theories.

Equality theory: the axiom system based on the three equality axioms.

After these rather lengthy preliminaries, the time has come to give my silly answer the more dignified status of a formula in the predicate calculus, which will take the form of a definition of the predicate Thing(t), read "t is a thing".

Thing defined: Thing(t) $\Leftrightarrow \exists x,y(\ x(t\neq y)\ )$. In words, for t to be a *thing* means that there is a *discriminator* x that can distinguish t from something else.

Is everything a thing? "Of course, what else could it be?" you say. But the axioms of relative identity theory turn out to be non-committal about this, i.e. they are consistent with either a *yes* or a *no*. It will be instructive to see just why this is so.

To say that everything is a thing means that everything satisfies the above-defined predicate Thing(t) – in symbols, $\forall t \exists x,y(\ x(t\neq y)$. How do we confirm that this formula is consistent with the relative identity axioms?

Here is where *models* come in. To show that a statement S in an axiom system *A* is consistent with the axioms of *A* it suffices to exhibit a *model* of *A* in a consistent axiom system *B* that makes S into a *theorem*. In fact, any *interpretation* of *A* in *B* in which S is a theorem will do; this interpretation needn't be a model, though models are often easier to work with. Here is a simple interpretation of relative identity theory in set theory that does the job for the yes answer:

Everything is a thing set model. Let x=y be set identity and define z(x=y) to mean x=y. Since it doesn't matter what z is, z(x=y) clearly satisfies the relative identity axioms.

To say that everything is a thing means $\forall t \exists x,y(\ x(t\neq y)\ )$. Since $x(t\neq y)$ means $t\neq y$, and there exist more than one set in set theory, this reduces to $\forall t \exists y(\ t\neq y\ )$, in words, for every set there is another set, which is indeed the case.

What about the NO answer? There are actually two NO answers: the weaker NO, which denies that everything is a thing, and the strong NO, which denies that there are any things at all. To show the consistency of the strong NO we don't actually need a model, since the three relative identity axioms can be directly deduced from it. Since the weaker NO is a consequence of the stronger NO, it too is consistent.

How do we know that the YES model above is actually a model and not an *abstract* interpretation? In the last section I defined a *model* as an interpretation in which the identity predicate of the system being modeled *matches* the identity predicate of its host. This definition calls for some amplification:

First of all, just what do we mean by the *identity predicate* of an axiom system *A*? The short answer is that it's the equality predicate in *A* that satisfies *Leibniz' law*:

Leibniz' law: Substituting equals for equals does not change truth values.

Substitutivity. Leibniz' law today is called *substitutivity.* For an equality "=" to be substitutive at x in Sx means that $x=x' \Rightarrow \forall y,z\ldots(Sx \Leftrightarrow Sx')$, where Sx is a sentence in which x, y, z… are free and Sx' is the sentence in which the variable x' replaces x.

Leibniz, like everyone else in his time and most people today, regarded the identity predicate as *absolute*, and he claimed that *universal substitutivity* is its defining feature. However, if we take identity to be relative, we can think of Leibniz' law as only defining substitutivity at one free variable.

Identity predicate of an axiom system *A*: the (unique) equality predicate that is substitutive at every free variable in the sentences of *A*. Note that this takes the form of Leibniz' universal substitutivity confined to the universe of *A*.

How do we apply this definition to an axiom system like relative identity theory which has no identity predicate among its primitives? The answer is that we *define* it in terms of whatever primitives there are. This definition has three steps. First we define what I'll call the *intrinsic identity* predicate at x. Second, we use this definition to define the intrinsic identity of a *predicate*. Third, we finally we *conjoin* the intrinsic identities of all primitive predicates into a single equality. This equality can be shown to be substitutive at every free variable in every sentence and is thus the identity of the axiom system.

Intrinsic identity at x. Suppose x is free in the sentence Sx. Define the *intrinsic identity predicate* at x to be the least refined equality predicate that is substitutive at x.

The fundamental theorem of intrinsic identity: The predicate defined by $x=x' \Leftrightarrow \forall y,z\ldots(Sx \Leftrightarrow Sx')$ is the intrinsic identity predicate at x in S, where y,z… are the other free variables in S.

Proof: First we note that "=" as defined above is indeed an equality predicate, which follows quickly from the fact that "⇔" satisfies the equality axioms for sentences. [17] We must then show that "=" as defined above is the least refined substitutive equality at x in S. Clearly it is substitutive, since $x=x' \Leftrightarrow \forall y,z\ldots(Sx \Leftrightarrow Sx')$ implies $x=x' \Rightarrow \forall y,z\ldots(Sx \Leftrightarrow Sx')$. Suppose there were a less refined equality, call it "≡", that is also substitutive at x, i.e. $x\equiv x' \Rightarrow \forall y,z\ldots(Sx \Leftrightarrow Sx')$. From $x=x' \Leftrightarrow \forall y,z\ldots(Sx \Leftrightarrow Sx')$ it follows that $\forall y,z...(Sx \Leftrightarrow Sx') \Rightarrow x=x'$. Therefore $x \equiv x' \Rightarrow x=x'$, which implies that "≡" is *not* less refined than "=", contradicting our supposition that a less refined substitutive equality exists. Ergo, "=" is it.

Let's define the intrinsic identity of a sentence to be the conjunction of the intrinsic identities of its free variables. The intrinsic identity of a predicate P is then defined as that of any sentence gotten by putting different free variables in all of its places. For instance, the intrinsic identity of "∈" is the intrinsic identity of the sentence $x \in y$.

Theorem: The intrinsic identity of an axiom system *A* is the conjunction of the intrinsic identities of its primitive predicates. [18]

This theorem tells us that we can define set identity $x=y$ in terms of the membership predicate $x \in y$ by the statement $x=x' \Leftrightarrow \forall y(\ (x \in y \Leftrightarrow x' \in y)\ \&\ (y \in x \Leftrightarrow y \in x')\ )$. It also tells us something that sounds almost paradoxical, which is that relative identity theory has an *absolute* identity predicate!

Quine, in his essay "The Scope of Logic", [19] presents a condensed version of the above construction of the intrinsic identity predicate of an axiom system, which he then uses as the basis for his claim that the concept of identity *belongs* to *logic*. We can grant him this, so long as we make it clear that the logical concept of identity, unlike other logical concepts like AND, OR and NOT, comes in an infinite variety of flavors. Any third-way interpretation of an axiom system has the same AND as its host, but not necesarily the same identity. Although Quine never uses the term 'relative identity", what he has really shown is that it's relative identity that belongs to logic. Incidentally, this same essay contains a very convincing argument for set membership *not* belonging to logic.

Now that we know what the identity of an axiom system is, we can return to the definition of a model, which was informally defined in Section 2 as an interpretation of an axiom system *A* in a host *H* in which the identity of *A matches* the identity of *H*. The third question, then, is what does it mean for two equalities to *match*?

The short answer is that they agree in their discriminations among everything that is not *Other*. Which brings us to the very important concept of *Other*, to which we must pay a fair amount of attention before we can get to matching.

So what is *Other*? As I said at the beginning of this section, *Other* is what doesn't matter. But this as it stands doesn't say very much, since there are many ways in which things can matter, and all degrees of mattering. But for a start we'll go to the black-and-white limit. It will help us to get us there to start out with the concept of a black-and-white equality, which I'll call a *division*.

Division: an equality that is subject to the additional axiom that there exist x and y that are unequal, and that if any z is unequal to either of them then it is equal to the other. In symbols, $\exists x,y(x \neq y)\ \&\ (\ \forall x,y,z(\ x \neq y \Rightarrow (z=y \text{ OR } z=y)\ )$.

Theorem. The intrinsic identity of a single-place predicate is a division.[20]

Section 4. What exists? What matters? What happens?

To exist: existere (to stand out)

Existence: t exists for x means ~x(t=x), abbreviated $t \in x$

World: The world of x (informal) means everything that exists for x.

Containment: x's world is contained in, y's world, written $x \subset y$, means that for all t, $t \in x \Rightarrow t \in y$.

Coextension: x and y have coextensive worlds, written $x \equiv y$, means that for all t, $t \subset x$ if and only if $t \subset y$.

What are existence and anti-existence?

Rectangular worlds and x and y coordinates.

http://www.gnosticchristianity.com/ch7.htm
William C. Kiefert

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[1] There is all too little sound historical knowledge about what Pythagoras actually said or thought, but this is one of the livelier stories about him.

[2] His definition was actually a bit more complex. He defined a *real* number as any *cut* in the rational numbers that divides them into an upper and a lower part. Our definition of the square root of two is the lower part of such a cut, which of course determines its upper part.

[3] Sometimes called an axiom *schema*.

[4] Instead of thinking of this as a meta-axiom, we can regard it as an infinity of axioms of the same kind, one for every definable single-place predicate

[5] In fact there are now a ggod many versions of Zermelo's set theory, but the differences among them won't concern us here.

[6] More exactly, we say that the set N is a *number* if there a set s such that for all m, m is a members of N if and only if m is equivalent to s.

[7] Here is why it leads to Russell's paradox. Consider the number 1, which is the set of all sets having one member. Now all reasonable sets theories agree that for every set s there is a unit set {s} of which s is the sole member. Thus the set of all sets with one member is in obvious 1-1 correspondence with the collection of all sets, whether or not that collection is itself a set. But it is again an axiom or theorem in all reasonable set theories that the union of all members of a set is also a set, which in the case of the number 1 is the set of all sets. The axiom of separation says that for any property P, including the property of not being a member of itself, there is a subset of any set that has P. Thus there must exist a subset of the set of all sets consisting of those that are not members of themselves, which is Russell's paradox..
Russell and Whitehead, in their heroic epic *Principia Mathematica*, attempted to get around this problem by stratifying sets in a way that would, so to speak, reproduce the concept of number on every level, but the resulting complexities have discouraged potential followers.

[8] Definition is not quite the right word; since the choice of the variable "x" is arbitrary; we can equally well substitute "Mr. B" for "the thing y that presides".

[9] The term *definite description* is sometimes used in the literature to mean what I am here calling a *definite noun phrase*. Whatever terms we use, it's essential to distinguish between these two linguistic forms.
[10] What Tarski calls a model should more properly be called a *relational* model.
[11] The word "matches" has a technical meaning that will be defined in Section 3.
[12] Define $x+y \equiv z$ to mean that the sum of the counters equivalent to x and y is the counter equivalent to z. This is not a functional predicate in set theory but it is meaningful for any sets x, y and z and becomes functional when we take ≡ to be Cantorian identity, which is its intrinsic identity predicate. Similarly for $xy \equiv z$.
[13] REWRITE Well, maybe it *is* circular, though in the dignified halls of higher mathematics we don't say "circular", we say "recursive". Strangely enough, it actually is a definition of sorts. Admittedly it doesn't tell us what a thing is *all by itself*; that's too much to ask. Rather it tells us what things are *wholesale*. It says that the criterion for the members of a plurality to be things is that you can tell them apart. Note that this is non-committal as to who or what does the telling apart. That is, it allows for different conceptions of *identity*, which in turn allows, by the above definition of things, for different conceptions of *thing-hood.* Thus it would seem that the third way, by relativizing identity, also *relativizes ontology*. As the mathematician Gian-Carlo Rota put it, "Identity precedes existence."

Actually, that's an over-statement. The truth is that existence in the material world almost always precedes identity. If you trip over a cobra in the dark and *identify* it as the garden hose, the magic word "as" won't save you. Much as you might like to turn the cobra into a *cobra as a garden hose* by choosing some more agreeable identity predicate, the cobra, who has no desire to become a garden hose, is not likely to collaborate.

Physical existence has a way of asserting itself that is indifferent to our powers of discrimination. The same is not true, however, of *abstract* existence, which, as its name implies, is abstracted from the flux of being by our powers of discrimination. Take the quality of *redness*, for instance. If we can't distinguish red things as such from other things, then redness as such doesn't exist for us; it's nothing. But if we *can,* then, by the above definition, redness does exist for us; it is something. Must red *physical* things exist in order for redness to exist? Or would redness still exist even if we only see red things in our dreams? These are subtle and difficult questions, but fortunately we are still in the simple-minded world of mathematics so, as in the case of John's brain transplant, we don't have to deal with them here.

Redness, and indeed any property, can be viewed as a certain way of *being the same*. If x and y are both red, which we'll write as Red(x) and Red(y), then they are the same color. Let's define $R(x=y)$ to mean that x and y are both red, where we'll take "R" to be an abbreviation of "AsRedThings".

[14] Buber regards the relationships of I-him and I-her as special cases of I-it. In this I strongly disagree. The ability to treat a *him* or a *her* as an *it* is the inability to empathize, which is what leads to most of cruelty and injustice in the world. Also, there are many other primary relationships of which *I* is a fragment, for instance *I-You* (plural), *I-us-them*, *I-we-it* etc.
[15] The other two are AND and NOT, though these can both be derived from NAND, i.e. NOT-AND, which is perhaps another example of a compound word that is more fundamental than its parts.
[16] For the sake of readability we are adopting here the common convention that when we assert a sentence containing free variables, it is implicit that these are universally quantified from the outside. Thus $x(y=y)$ is shorthand for $\forall x,y(\ x(y=y)\ )$ etc.
[17] ... spell out the details.
[18] Proof ...
[19] ref
[20] proof